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OPTIMALITY AND STRONG STABILITY OF CONTROL SYSTEMS*

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1. Introduction.

A necessary attribute of any real control system is that it be stable under perturbations, and the oldest method of designing feedback control systems is based upon making the desired state asymptotically stable in the linear approximation. This dates back to J. C. Maxwell[1] in 1868 and J. Vyshnegradskii[2] in 1876. In more recent optimal control theory it is well known for infinite time optimal control that the desired state will be asymptotically stable if the integrand of the performance functional is positive definite. Examples are also known of some special control systems, which reduce the error in control to zero in finite time, that have a "strong stability".[3] In general, however, there is very little known about stability under perturbations of optimal control systems particularly when the control is over a finite period of time and the control as a function of the state of the system has discontinuities. Systems which are designed to reduce the error in control to zero in minimum time behave badly (e.g., chattering) when the error is small due to time delays in switching and other perturbations. Thus near the desired state the system is often designed to switch from optimal control to linear control.

In this paper, results are presented which indicate that considerable improvement in performance can be expected by designing the system to be time optimal to a small neighborhood of the desired state rather than designing it to be optimal to the desired state itself. We restrict ourselves here to normal autonomous linear time optimal control systems[4] with the objective to reach a small ball around the origin (to achieve a small error in control) in minimum time. We are then able to show that this time optimal control has a strong stability under perturbations and is in a certain sense the "best" of all "stabilizing controls".[5]

The theory indicates the advantages of this time optimal control should be that 1) the neighborhood where the optimal control behaves badly should be smaller, 2) the time to reach this neighborhood is a minimum, 3) outside the neighborhood the stability under perturbations is stronger, and 4) the computation of optimal control is easier because of the additional transversality condition.

2. Optimality.

The mathematical model for the control system is $(\dot{x} = \frac{dx}{dt})$

$$\dot{x} = Ax + Bu \tag{1}$$

where the state of system x is an n-vector, u is the control function and is an r-vector, A is a constant $n \times n$ matrix, and B is a constant $n \times r$ matrix. We consider first of all the class

Ω of admissible open loop controls $u(t)$ with the property that u is measurable on finite intervals of $[0,\infty)$ and is limited in magnitude by $|u_i(t)| \leq 1$, $i = 1,\ldots,r$. The target is the ball $\mathscr{B} = \{x;\ |x| \leq \epsilon\}$ of radius ϵ about the origin and $S = \{x;\ |x| = \epsilon\}$. We assume that the system is normal, which implies that the time optimal control to hit $\mathscr{B}$ is for each initial state $x(0) = x^o$ unique and bang-bang.[4]

Let $T(x)(x \notin S)$ be the minimum time to go from x to $\mathscr{B}$ and define $\Sigma(t) = \{x;\ T(x) = t\}$, $t \geq 0$. The set $\Sigma(t)$ is an isochrone. It is then not difficult to see that

i) $\Sigma(t)$ is the boundary of a strictly convex compact set $\mathscr{A}(t)$ for each $t > 0$.

ii) If $x^o \in \Sigma(t)$ and the optimal control from x^o to $\mathscr{B}$ hits $\mathscr{B}$ at v, then v'^{-At} is an outward normal to $\mathscr{A}(t)$ at x^o and $\Sigma(t)$ is differentiable at x^o (has a unique support hyperplane at x^o).

It can then be shown that

Theorem 1. <u>On its domain of definition</u> $T(x^o)$ <u>is continuously differentiable</u>.

3. Strong Stability.

We want to define now as large a class Φ of admissible feedback controls $\varphi(x)$ as we can which satisfy $|\varphi_i(x)| \leq 1$ for $i = 1,\ldots,r$. Since for x^o outside $D = \bigcup_{t>0} \mathscr{A}(t)$, there is no admissible (open loop) control $u(t)$ that brings (1) within D, we confine ourselves to D. We will say that $\varphi \in \Phi$ if in some sense there is for each $x^o \in D$ a uniquely defined solution $x(t)$ of

$$\dot{x} = Ax + B\varphi(x) \tag{2}$$

for each $x^o \in D$ for as long as $x(t) \in D$ $(t > 0)$ and which is such that $u(t) = \varphi(x(t))$ is an admissible open loop control $(u \in \Omega)$. The time optimal feedback control $\varphi^*(x)$ obtained by synthesizing the optimal open loop control $u^*(t)$ is clearly an optimal feedback control. It is then rather easy to show, from the above, that this optimal control has the following strong stability property.

Up to now we have suppressed dependence on ϵ. Taking this into account we replace $\mathscr{A}(t_1)$ by $\mathscr{A}(t_1,\epsilon)$ and $\mathscr{B}$ by $\mathscr{B}(\epsilon)$. Consider the perturbed system

$$\dot{x} = Ax + B\varphi^*(x) + p(t,x). \tag{3}$$

Then

Theorem 2. *Given $t_1 > 0$ and $\epsilon > 0$ there exists $\rho(t_1,\epsilon)$ such that if $|p(t,x)| \leq \rho_1 < \rho(t_1,\epsilon)$ for all $t \in [0,\infty)$ and all $x \in \mathscr{A}(t,\epsilon)$, then for some $T(\rho_1)$ each solution of (3) starting in $\mathscr{A}(t_1,\epsilon)$ reaches $\mathscr{B}(\epsilon)$ in time less than $T(\rho_1)$.*

As with an asymptotically stable equilibrium it can happen that $\rho(t_1, \epsilon) \to 0$ as $\epsilon \to 0$. However, here the time to reach $\mathscr{B}(\epsilon)$ approaches t_1 as $\epsilon \to 0$ and $\mathscr{A}(t_1, 0) \subset \mathscr{A}(t_1, \epsilon)$ for all $\epsilon > 0$. For a normal system $\mathscr{A}(t_1, 0)$, the attainable set to the origin in time t_1, is strictly convex (and hence contains the origin in its interior) but its boundary will, in general, not be smooth.

The general principle behind this result on strong stability applies to much more general situations and we have presented here the simplest possible case.

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[1] Mr. J. C. Maxwell on Governors, Proc. Royal Society London 16(1868), 270-283.

[2] Vyshnegradskii, J., Sur la théorie générale des régulateurs, Compt. Rend. Acad. Sci. Paris 83(1876), 318-321.

[3] See, for instance, LaSalle, J. P., Stability and control, J. SIAM Control, Ser. A, 1(1962), 3-15. In this paper it was shown when $r = n$ and B is nonsingular that there is a control with strong stability. These restrictions are severe and usually are not satisfied by real systems.

[4] See LaSalle, J. P., The time optimal control problem, Contrib. to Theory of Nonlinear Oscillations, V, 1-24, Princeton Univ. Press, Princeton, N. J., 1960. Throughout we follow the notations of this reference.

[5] This is related to a concept of a "best stabilizing control" introduced by P. Brunovsky. On the best stabilizing control under a given class of perturbations, Czech. Math. J. 15(1965), 329-369. The existence of a "best stabilizing control" was shown in this paper when $n = 2$ and $r = 1$. Brunovsky has since proved this in the general case (private communication).